Received: *August 8, 2000*, Accepted: *August 18, 2000*

Deformation quantization as the origin of D-brane non-abelian degrees of freedom

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Abstract: I construct a map from the Grothendieck group of coherent sheaves to K-homology. This results in explicit realizations of K-homology cycles associated with D-brane configurations. Non-abelian degrees of freedom arise in this framework from the deformation quantization of N-tuple cycles. The large-N limit of the gauge theory on D-branes wrapped on a subvariety V of some variety X is geometrically interpreted as the deformation quantization of the formal completion of X along V.

Keywords: D-branes, M(atrix) Theories.

Contents

1. Introduction

Soon after Polchinski's identification of D-branes as the nonperturbative objects in perturbative string theory that carry Ramond-Ramond charge [1], Witten [2] argued that the low-energy effective action for the dynamics of nearly coincident D-branes should be appropriate dimensional reductions of supersymmetric Yang-Mills (sYM) theory in ten dimensions. An exciting aspect of this identification [2] was the fact that the transverse spacetime coordinates describing coincident D-branes metamorphosed into hermitean matrices, the adjoint scalars in the dimensional reduction of sYM theory. Witten's argument was based on supersymmetry, specifically the BPS property of parallel D-branes. This appearance of non-abelian structure in spacetime coördinates was the first indication in the context of a perturbatively consistent theory of quantum gravity that spacetime might have some intrinsic non-commutativity.

Following seminal work by Sen [3] on the construction of non-BPS D-branes as solitons on the world-volumes of D-brane–anti-D-brane pairs, and other earlier work [4, 5], Witten [6] suggested that D-brane charge should be classified by the compactly supported elements in the K-theory of spacetime. Since K-theory deals with vector bundle equivalences, these constructions provide hints of non-abelian degrees of freedom in D-brane physics, but no connection to deformation quantization. On the other hand, Moore and Witten [7] argued that Ramond-Ramond fluxes should be classified by K-theory with an appropriate suspension of spacetime, an interpretation which received strong independent support [8].

Finally, following pioneering work of Connes, Douglas and Schwarz [9], Seiberg and Witten [10] showed that non-commutativity appeared in open string theory via a deformation quantization interpretation of the effects of a large constant B-field on the worldvolume of a D-brane. This non-commutativity seems to have nothing to do with the non-abelian character of the gauge theories relevant for D-brane physics mentioned above, though [11] finds an intriguing connection in the presence of a large B-field.

This 'historical' sketch is an attempt to set the stage for the present paper. I argued in [12] that D-brane charges should take values in K-homology [13, 14, 15] instead of K-theory essentially based on covariance with the original description of D-brane physics [1]. I suggested KK-theory as the appropriate framework for classifying configurations with both Ramond-Ramond charges and fields present.

The aim here is to show how non-abelian degrees of freedom arise from deformation quantization by linking D-brane charges to K-homology [12].

2. Outline

In this section I outline the argument for the connection mentioned above. Requisite details are given in later sections.

I first construct a map from the Grothendieck K-group of coherent sheaves to K-homology. (Sharpe [16] was the first to note the relevance of the K-theory of coherent sheaves in the context of D-branes.) The first fact needed is that coherent sheaves are pushed-forward under morphisms of varieties. The coherent sheaf K-groups, which we denote $G^*(A)$, where A is the ring of interest, transforms covariantly under morphisms of varieties and as such we expect that there should be a map from $G^*(A)$ to the K-homology of the variety, just as there is a dual map from the K-theory of locally free sheaves (equivalent to algebraic or analytic vector bundles) to the topological K-theory. I have not been able to find a direct treatment of this map from coherent sheaves to K-homology cycles in the mathematics literature, so I will give a direct construction later. The map from projective resolutions of structure sheaves of subvarieties to K-theory is, of course, well-known.

The important point about coherent sheaves is that they are supported on algebraic subvarieties. Thus, for example, on the real line considering the ring of polynomials R and the ideal of functions that vanish at $x = 0$ to order N, $R/(x^N)$ is a coherent sheaf on the real line which vanishes everywhere except for the stalk over $x = 0$. This brings us to our second fact: Merkulov [17] showed that the deformation quantization of the function algebra associated to this 'N-tuple point' is the algebra of $N \times N$ matrices. The deformation parameter of physical interest is real, not imaginary as is the mathematicians' wont, which results in the appropriate algebra being the complex $N \times N$ matrices, with the physically appropriate real section being that of hermitean matrices.

Since the appearance of Merkulov's paper, it has been a puzzle as to how this calculation fits in with D-brane physics, since there was no context in which D-branes were naturally described in terms of algebraic equations defining subvarieties. Now, finally, we bring these facts together by noting that if, as argued in [12] on grounds of covariance, D-brane charge is classified by K-homology, there is a direct link between the deformation quantization of coherent sheaves associated to K-homology cycles and the appearance of non-abelian degrees of freedom on D-brane worldvolumes. I should note that this is not the application suggested in [17].

According to the construction given here, there is a canonical way of computing the operator algebra associated with a given D-brane configuration on any algebraic variety. The matrix model description of intersecting branes on a variety is an especially interesting special case [18]. In the case that D-brane worldvolumes are non-compact (as is usual for Dp-branes ($p \geq 0$) in Minkowski space, for example), a more appealing formalism for describing D-branes is in terms of a family (parametrized by the worldvolume) of transverse K-homology cycles. The geometry of the large-N limit is that of a deformation quantization of the formal completion of the ambient variety along the support of the coherent sheaf. These topics are discussed in the last section.

Now for the details with no pretence at mathematical sophistication.

3. Coherent sheaves to K-homology

Ref. [16] reviewed the Grothendieck constructions in the context of K-theory and D-branes [6], but the emphasis was on smooth projective varieties where Poincaré duality gives an isomorphism between the groups $K_*(A)$ and $G^*(A)$. In the physically interesting situation of non-compact manifolds there is no canonical isomorphism between these groups. The non-compact case leads to another motivation for identifying D-brane charge with K-homology: Recall that the homology of singular chains and de Rham cohomology with compact supports are isomorphic, and that the duals of compactly supported de Rham cohomology groups are isomorphic to de Rham cohomology groups [19] without assumptions of finiteness or compactness. According to Witten [6], D-brane charge is also most naturally measured in terms of compactly supported (in transverse directions) K-theory classes, so one can again argue for K-homology as the appropriate group.

Before giving the construction of K-homology cycles from coherent sheaf data, I recall the definition of these cycles [15, 20]. (A more compact form of the following definition would be in terms of $\mathbb{Z}_2$ graded objects.) A cycle in $K^0(A)$, the K-homology group of the algebra A, is specified by two Hilbert spaces $\mathcal{H}_{0,1}$, a linear map $\Theta : \mathcal{H}_1 \to \mathcal{H}_0$, and representations ϕ_i of A on $\mathcal{H}_i, i = 1, 2$. These are linked by requiring that for any a in A, the operators

$$\phi_0(a)\Theta - \Theta\phi_1(a)\,, \qquad \phi_0(a)(\Theta\Theta^* - 1)\,, \qquad \text{and} \qquad \phi_1(a)(\Theta^*\Theta - 1) \tag{3.1}$$

are all compact operators. For orientation, on a non-compact manifold X a possible choice for A is the algebra of continuous bounded functions that vanish at infinity,

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$C_0(X)$, and the Hilbert spaces could be the spaces of L^2 sections of two vector bundles $E_i, i = 0, 1$ on X, with the obvious action of continuous functions on sections of vector bundles by pointwise multiplication. It is important in this non-compact context to notice that we are not assuming that Θ is Fredholm.

A coherent sheaf is a sheaf that has a projective resolution by locally free sheaves [21]. This means there is an exact sequence

$$\cdots \to \mathcal{F}_n \to \cdots \mathcal{F}_0 \to \mathcal{S} \to 0\,, \tag{3.2}$$

where the sheaves $\mathcal{F}_i$ are sheaves of sections of vector bundles. Coherent sheaves over varieties therefore have the possibility of being compactly supported, since the dimension of the stalks of $\mathcal{S}$ is related to the rank of the linear maps between $\mathcal{F}_i$. We therefore compute the values of determinants of submatrices of these maps, and the zeros of these determinants determine the support of $\mathcal{S}$. (Coherent sheaves can be defined much more generally, not just on analytic or algebraic varieties. Their description in terms of finitely generated modules defined by a finite set of relations is appropriate in general [22]).

An alternative local characterization of a coherent sheaf that we will use is the following: Locally a coherent sheaf has a projective resolution by free modules

$$\mathcal{O}^n \xrightarrow{\sigma} \mathcal{O}^m \to \mathcal{S} \to 0\,, \tag{3.3}$$

where $\mathcal{O}$ is the sheaf of (germs of) holomorphic functions. These local resolutions do not necessarily fit together globally, hence the need for the general characterization given in (3.2). Now, by extending with the identity map, we can take (3.3) to be of the form

$$\mathcal{O}^\infty \xrightarrow{\hat{\sigma}} \mathcal{O}^\infty \to \mathcal{S} \to 0\,, \tag{3.4}$$

where we assume that $\hat{\sigma} - 1$ is of finite rank.

We now go out of the algebraic or analytic category and consider a partition of unity subordinate to these local projective resolutions (3.4). Since we want to construct a map to K-homology, we expect to lose the algebraic structure in any case, so this is acceptable. By polynomial approximation in each patch, we extend $\hat{\sigma}$ to smooth sections. Finally we recall that Hilbert bundles are trivial since the infinite unitary group is contractible so we encounter no obstructions in patching these local projective resolutions together to obtain a global resolution of $\mathcal{S}$ by sheaves of sections of Hilbert bundles E_i

$$\Gamma(E_1) \to \Gamma(E_0) \to \mathcal{S} \to 0\,, \tag{3.5}$$

picking arbitrary hermitean metrics on the fibres. We restrict ourselves now to a coherent sheaf $\mathcal{S}$ such that the support of $\mathcal{S}$ is strictly contained in the variety X. Then $\hat{\sigma}$ extends to a map Θ from the Hilbert space of L^2 sections of E_1 to the Hilbert space of L^2 sections of E_0 which is almost everywhere the identity. Since the whole construction is local, the action of Θ commutes with that of pointwise multiplication by functions in $C_0(X)$. Finally while we do not expect $\Theta\Theta^* - 1$ or its adjoint to be compact for X non-compact, we do expect (3.1) to be satisfied with the restriction that the support of $\mathcal{S}$ is strictly contained in X.

It seems straightforward to argue that the map from coherent sheaf data to K-homology data we have just constructed is well-defined in going to K-homology since we allow ourselves the use of smooth homotopies even though a short exact sequence of coherent sheaves

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0 \tag{3.6}$$

does not necessarily split algebraically or analytically. Thus we have finished constructing a map from coherent sheaves to K-homology, and can therefore think of K-homology cycles as being associated with subvarieties in X that arise as the support of coherent sheaves. This is more data than just specifying a subvariety as a homology cycle.

4. Deformation quantization of coherent sheaves

I turn now to a brief recapitulation of Merkulov's calculation [17]. Consider the N-tuple point, the solution of the equation $x^N = 0$ on the affine real line. The cotangent bundle has a canonical symplectic structure so the complexified algebra of functions on this phase space, extended to formal power series in the deformation parameter, can be deformed via the Moyal product [23]:

$$f * g(x,p) \equiv \exp\left(\frac{i\hbar}{2}\left[\frac{\partial^2}{\partial p \partial \tilde{x}} - \frac{\partial^2}{\partial \tilde{p} \partial x}\right]\right) f(x,p) g(\tilde{x},\tilde{p})\bigg|_{(x,p)=(\tilde{x},\tilde{p})} . \tag{4.1}$$

The obvious projection π from the phase space to the real line leads to a pullback of the ideal generated by x^N in the algebra of functions on the line. The ideal I_N in the deformed algebra consists of elements of the form $g(x,p) * \pi^*(x^N)$. The normalizer $\mathcal{N}_N$ of this ideal is

$$\mathcal{N}_N \equiv \left\{ f(x,p) : \pi^*(x^N) * f \text{ is in } I_N \right\} . \tag{4.2}$$

The point of introducing the normalizer is that I_N is a two-sided ideal inside $\mathcal{N}_N$. We can therefore construct the quotient algebra. Finally, by explicit computation, Merkulov showed that $\mathcal{N}_N / I_N$ is isomorphic to the algebra of $N \times N$ complex matrices, with the induced product in the quotient algebra mapping into matrix multiplication.

A remarkable fact about this construction is that the induced product is meromorphic in $\hbar$. The reason for this is interesting, since naïvely one might have supposed that the functions on phase space would be generated by monomials of the form $x^i p^j, 0 \leq i, j \leq N-1$. This is not however the case and there are terms with

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higher powers of p present. For example, the general element in the quotient algebra $\mathcal{N}_2/I_2$ is [17]

$$h \equiv a + b\left(p + \frac{i}{\hbar}p^2 * x\right) + (c + dp) * x\,, \tag{4.3}$$

with a, b, c, d complex numbers. By dimensional analysis, the higher power of p in (4.3), $bp^2 * x$ must be accompanied by an appropriate negative power of $\hbar$. In other words, the N-tuple point is a well-behaved quantum system with no classical phase space Poisson algebra. The reader surprised by the appearance of terms singular as $\hbar \downarrow 0$ may wish to look at other examples [24].

Merkulov's construction is much more general than the example I have reviewed. As I mentioned earlier, the difficulty in utilizing this construction in matrix models of D-brane dynamics arises because *a priori* there is no reason to associate D-branes with solutions of algebraic equations. However, in light of our association of D-brane charge with K-homology, and the construction above of K-homology cycles from coherent sheaves, we see a natural application of Merkulov's idea. Namely, the deformation quantization of the coherent sheaf associated with a K-homology cycle is the operator algebra of interest.

K-theory is invariant under deformation quantization [25], a fact used in [26] to understand D-brane phase transitions in K-theory. It is reasonable to expect then that the homology theory dual to K-theory, i.e. K-homology, will also be invariant.

To formalize the deformation quantization version of the K-homology cycle, we now take $\mathcal{H}_i \equiv L^2(T^*\mathbf{R}) \cap \mathcal{N}_N, i = 0, 1$ and the operator Θ to be an appropriate bounded version of multiplication by x^N, for example:

$$\Theta : f(x,p) \mapsto \pi^*\left(\frac{x^N}{(1 + x^{2N})^{1/2}}\right) * f(x,p)\,. \tag{4.4}$$

(The definition of $\mathcal{N}_N$ in (4.2) needs appropriate modification of course.) The algebra of interest in this case is $A \equiv C_0(\mathbf{R})$. We need to check that the representation of this algebra on $\mathcal{N}_N$ is actually a representation, in other words for all a in A and f in $\mathcal{N}_N$, $\pi^*(a) * f$ is in $\mathcal{N}_N$. This is trivially verified using the associativity of the $*$ product and the definition of $\mathcal{N}_N$. Finally this representation of A on $\mathcal{H}_i$ commutes with the operator Θ.

5. Superconnections, families and formal completions

There is, of course, more to D-branes than just the appearance of matrix degrees of freedom. It is easy to see that an N-tuple point on a plane, for example, will lead to an algebra of two independent matrices and so on. Thus coïncident D(-1)-branes are easily described in this framework. Further, for a D-brane wrapped on a cycle in a compact variety, we get an operator algebra description that is not described in terms of matrices parametrized by a worldvolume, but instead directly in terms of an operator algebra *including* the worldvolume [18].

For a non-compact worldvolume, it is more appropriate to think of a family, parametrized by the worldvolume, of operator algebras describing the transverse geometry. The operator Θ is now parametrized by the worldvolume and is combined in a superconnection with a connection $\nabla : \mathcal{H}_i \to \mathcal{H}_i$. It is a great deal simpler to work throughout with a $\mathbb{Z}_2$-graded formalism as set out in [29]. This would lead us quite far from the main point of this paper so I will address this elsewhere.

I want to briefly address perhaps the most interesting aspect of this formalism, which is to relate the coupling of Ramond-Ramond fields to D-brane worldvolumes to the pairing between K-theory and K-homology. Just as K-theory classes are represented as cohomology classes via the Chern homomorphism, algebraic K-theory classes are represented as cyclic homology cycles and K-homology cycles are represented via the dual Chern homomorphism as cyclic cohomology cocycles [27]. To get an intuitive idea for what is involved in this, recall that

$$\mathcal{L}_v = \mathrm{d}\iota_v + \iota_v \mathrm{d}\,. \tag{5.1}$$

The Chern-Simons coupling as usually described in the literature [28] takes the form

$$\int_V \operatorname{tr}\exp(F + \iota_\phi^2)\phi^* C\,, \tag{5.2}$$

where (abusing notation) $\phi^* C$ is the pullback of the spacetime Ramond-Ramond potential to the worldvolume of the D-branes with ϕ describing the fluctuation of the D-brane worldvolume in the transverse directions (in the absence of a B-field). Computing $(D + \iota_\phi)^2$, where $D = \mathrm{d} + A$, we find

$$(D + \iota_\phi)^2 = F + \iota_\phi^2 + \hat{\mathcal{L}}_\phi\,, \tag{5.3}$$

where $\hat{\mathcal{L}}$ is a gauge covariant Lie derivative. The integral in (5.2) is over the worldvolume V so we can try to think of it as

$$\int_V \operatorname{tr}\exp(F + \iota_\phi^2)\exp(\hat{\mathcal{L}}_\phi)C \tag{5.4}$$

and hence in the form

$$\int_V \operatorname{tr}\exp((D + \iota_\phi)^2)C\,, \tag{5.5}$$

where we have ignored the fact that the operators involved do not commute. Nevertheless, the correct form of (5.5) appears to be the JLO-cocycle as represented in [30].

Finally, a geometric picture of the large-N limit emerges from the representation of D-branes given in this paper. Recall a standard construction in algebraic geometry: If a subvariety is defined by the vanishing of some ideal of functions I then the N^{th} infinitesimal neighbourhood of a subvariety is defined by the vanishing of I^N. The formal completion [31] of these infinitesimal neighbourhoods is analogous to the construction of p-adic numbers from the integers. Thus, the geometric meaning: The large-N limit of D-brane physics is the deformation quantization of the formal completion.

Acknowledgments

I am indebted to S. Gukov and R.C. Gunning for enlightening discussions. In particular, Gukov pointed out the relevance of Sharpe's work. I am grateful to G. Moore and J. Rosenberg for helpful remarks. This work was supported in part by NSF grant PHY98-02484.

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